

ANALYTICAL SOLUTION OF ORDER 15 AND ORDER 7 FOR INITIAL VALUE PROBLEMS INVOLVING RATIONAL INTEGRATOR

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Abstract

This paper is based on a class of rational integrator to establish the region of absolute stability of an order 15 and 7 rational integration, both integrators are constructed from $s(\bar{h}) = F(R, \theta)$ where $F(R, \theta)$ is a complex function. The rational integrating function (RIF) was expanded successfully using polar analysis. The Jordan curve is constructed with the help of MATLAB packages. Furthermore, it was discovered that the rational instability (RIS) are on the positive side of the complex planes of both integrators, while the region of absolute stability was outside the curve. Finally, it was further established that rational integrators fall within the interval $-61.48147892 < \tau < 61.48148092$ and $-19.2 < \tau < 19.2$ where τ is the encroachment point. It can be said therefore, that both rational integrators formula are A-stable.

Keyword: Absolute Stability, Encroachments Point, Binomial Theorem, Complex function and Polar Analysis .

1.0 Introduction.

Numerical methods for systems of Ordinary Differential Equations (ODEs) have been attracting much attention because they enhance the solution of problems arising from the mathematical formulation of physical situations in chemical kinetic and population models. Numerical solution of ordinary differential equations can be obtained using method such as Rational Integrators, linear multistep method, Runge-kutta method and Exponential method. The overall aim of the study is to derive the rational integrator method. The specific objective will be to determine the stability of the method, encroachment point and implement the formula on initial value problem of the form $y' =$

$f(x,y)$, $y(x_0) = y_0 \forall a \leq x \leq b$ as well as draw the Jordan curve of $S(\bar{h}) = F(R,\theta)$ function derivable from the formula, in order to establish the region of stability

There are quite a number of problems that are stiff, they are mainly from reaction, chemical kinetic and life sciences. According to Aashikpelokhai(1991, 2000), Momou(2006), Lambert(1973, 2000), Corless(2001), Fatunla(1980, 1982, 1988), Abbulimenet. al (2007) and Agbeboh (2007) who established the region of absolute stability for $k = 13$.

The problem is to find a numerical solution to the IVP which is represented by $y' = f(x,y)$: $a \leq x \leq b$ where $f(x,y)$ is defined and continuous in a region $D \subset [a,b]$ that is stiff.

Aashikpelokhai(2011) worked on general $[L \leq 0, M \geq 0]$ one integer is proposed for the solution of the initial value problem in ordinary differential equation. It is general to the extent that the natural number L, M , representing the numerator and denominator degrees of the integrator can be selected randomly without any prior knowledge. Sample tests carried out reveal favourable results. Also, Aniral (2015) worked on a comparative study on numerical solution of initial value problems (IVP) for ordinary differential equation (ODE) with Euler and Runge Kutta methods, he presents Euler method and fourth-order Runge Kutta method (RK4) for solving initial value problems (IVP) for ordinary differential equation (ODE). The two proposed methods are quite efficient and practically well suited for solving those problems. In order to verify the accuracy, he compares numerical solutions with the exact solution. Numerical comparison between Euler method and Runge Kutta method have been presented they also compare the performance and the computational effort of such methods in order to achieve higher accuracy in the solution the step size needs to be very small. Finally they investigate and compute the error of the two proposed methods for different steps sizes to examine superiority several numerical examples one given to demonstrate to rehabilitate and efficiency.

Agbeboh and Omonkar(2010) studied the solution of singular initial value problems in ordinary differential equations using a new third order inverse Runge-Kutta method and state that A new third order rational Runge-Kutta formula is derived and implemented results obtained are compared with the existing formulae to determine the level of its performance, consistency and accuracy. It was discovered from the computation carried out that errors are minimized while using the new scheme. According to Agbeboh et al.(2020) in their work, “on the Stability Analysis of Rational Integrator Method for the solution of Initial Value Problems in Ordinary Differential Equation”, they concluded that the Region of Absolute Stability (RAS) of the Rational Interpolation method lies entirely on the Left- hand of the complex plane. At the same time, the region of instability (RIS), lies within the Jordan Curve. Their work revealed that the Region of Absolute Stability (RAS) for $K = 8$ is a superset of the entire left- hand of the complex plane. Finally, it was established that the encroachment point τ , lie within the interval ± 140.6 and the encroachment point is visible from the corresponding values of $\emptyset$ and R . Furthermore, the rational

interpolation scheme is not only A- stable but also L-stable. Agbeboh(2014) has it that in numerical analysis, the smallness of the individual error is called accuracy but the ability to keep the effect of this error under control is called stability.

Agbeboh(2007) worked on implementation of a new 4th order Runge-kutta formula for solving Initial Value Problems (IVPs) which successfully produced result that compared favorably well with other existing method. All of these works were basically for the improvement of numerical solutions to initial value problems giving us room for our study. The rational; interpolating method

are used to find numerical solution to rational ODEs of the form. $\frac{P_n(x)}{Q_m(x)}$ where both P and Q are

ODE polynomial form, the result are better solved by using pade-approximation. Our concern here is the analysis and investigation of the derivation and the implementation of order 15 and 7 rational integrator for the solution of IVPs in ODEs.

Ebhohumen et al. (2020) studied one-stop rational integrators method for solving initial value problems (IVPs) in ordinary differential equations, and designed one-stop rational integrators (OSRIs) of order 2k-1 method for solving initial value problems (IVPs) in ordinary differential equation (ODEs). The method was implemented and compared with other existing methods to ascertain the level of performance, in terms of results produced from solving some initial value problems. Problems involving modeling of real-life situation in an ordinary differential equation were handled by their method.

Ehiemua and Agbeboh (2020) worked on the stability of order 7 rational interpolation scheme for solving initial value problems in ordinary differential equation and they stated that the validity of a method can be verified by analyzing the stability property of a rational interpolation scheme and they carried out investigation on a rational interpolation scheme of order 7 through two major processes of converting the resulting rational function to a complex outlook and then transforming the complex function to a polar form, from which the stability region of the method is constructed in the form of Jordan curve. The regions of stability and instability as well as the encroachment interval of the scheme are determined with the use of Maple 18 and Matlab package.

Elakhe and Aashikpelokai (2013),also studied singularoscillatory – stiff rational integrator and they derive a general singular oscillatory – stiff rational integrator of order (s + 3) for s = 0, 1, 2, 3, 4... For the solution of initial value problems in ordinary differential systems that are singular oscillatory or stiff. They compared out integrators with certain maximum order second derivative hybrid multi – step methods, certain Tau and Euler methods the adaptive implicit and classical Runge – Kutta methods and some existing conventional methods the result show good improvement over the existing methods compared with the stated that, the region of absolute stability (RAS) of the rational interpolation method lies entirely on the left – half of the complex

plane. Therefore, a rational interpolation method that is both A – stable and L – stable have produced an efficient algorithm for the solution of IVP in ODE.

Elakhe and Aashikpelokhai (2011) worked on efficient order seven and eight rational integrators, they derive two new numerical integrators of order seven and eight for the solution of initial value problems in ordinary differential systems that are stiff, singular or oscillatory, they also compare the integrator with existing method, their results show good improvement over known existing method.

2.0 Methods

Basically, there are two ways of approaching the expansion of the integrating function, which are the $F(u,v)$ and $F(R,\theta)$ methods, one leads to the other. Using the binomial expansion $F(u,v)$ to establish the complex integrating function, then reduce the expansion to polar form, and introduce $F(R,\theta)$, where R represents the radius of the complex integrator while argument (θ) is the angle of rotation. The expansion of the resulting $F(u,v)$ functions is carried out in complex form by using De Moivre method. After the expansion, the integrators are subject to a sequence of salability analysis to establish the stability and the region of absolute stability (RAS). The stability function is as stated below and expanded as follows:

$$S(h) = \frac{\sum_{i=0}^{k-1} (2k-1-i) \binom{k-1}{i} \eta^i}{\sum_{i=0}^k (-1)^i (2k-1-i)! \binom{k}{i} \eta^i} \quad (2.1)$$

$$\psi(\eta) = \sum_{i=0}^k (-1)^i (2k-1-i)! \binom{k}{i} \eta^i$$

$$\theta(\eta) = \sum_{i=0}^{k-1} (2k-1-i)! \binom{k-1}{i} \eta^i$$

$$S(\eta) = \frac{\theta(\eta)}{\psi(\eta)} < 1$$

Which holds if and only if

$$|\phi(\bar{h}) < \psi(\bar{h})|$$

If and only if

$$|\phi(\bar{h}) - \psi(\bar{h})| < 0$$

Since $\bar{h} = u + iv$ and $i = \sqrt{-1}$

$$F(u,v) = |\phi(\bar{h}) - \psi(\bar{h})|$$

$\therefore S(\bar{h}) < 1$ if and only if $F(u,v) < 0$.

Working in line with Aashipelokhai (1991) a class of Non-linear one-step rational integral is established that will be widely used in the solution of IVPs in ODEs from a historical outline. Our task is to establish the stability function for the rational integrator for the case $k = 8$

$$S(\bar{h}) = \frac{\sum_{i=0}^{k-1} (2k-1-i)! \binom{k-1}{i} \bar{h}^i}{\sum_{i=0}^k (-1)^i (2k-1-i)! \binom{k}{i} \bar{h}^i}$$

$$S(\bar{h}) = \frac{\theta(\bar{h})}{\psi(\bar{h})} \leq 1 \text{ where } \theta(\bar{h}) = \sum_{i=0}^{k-1} (2k-1-i)! \binom{k-1}{i} \bar{h}^i$$

$$\psi(\bar{h}) = \sum_{i=0}^7 (-1)^i (2k-1-i)! \binom{k}{i} \bar{h}^i$$

$\forall k = 8$, such that

$$\frac{\sum_{i=0}^7 (2(8)-1-i)! \binom{7}{i} \bar{h}^i}{\sum_{i=0}^8 (-1)^i (2(8)-1-i)! \binom{8}{i} \bar{h}^i} = \frac{\sum_{i=0}^7 (15-i)! \binom{7}{i} \bar{h}^i}{\sum_{i=0}^8 (-1)^i (15-i)! \binom{8}{i} \bar{h}^i} \quad (2.2)$$

Now we seek $|\phi(h)| < 1$ which holds if and only if

$$\left| \frac{\phi(h)}{\psi(h)} \right| < 1$$

If and only if

$$|\phi(h)| < |\psi(h)|$$

If and only if $|\phi(h)|^2 - |\psi(h)|^2 < 0$

By a process of binomial expansion, we expand the numerator and the denominator as follows

$$\theta(\bar{h}) = \sum_{i=0}^7 (15-i)! \binom{7}{i} \bar{h}^i$$

$$\text{And } \psi(\bar{h}) = \sum_{i=0}^8 (-1)^i (15-i)! \binom{8}{i} \bar{h}^i$$

$$\text{With } \binom{k}{i} = \frac{k!}{i!(k-i)!}$$

From the numerator we have $\binom{7}{i} = 7(7-1)(7-2) \dots (7-i+1) = \frac{7!}{i!(7-i)!}$

And $\theta(\bar{h}) = \sum_{i=0}^7 (15-i)! \binom{7}{i} \bar{h}^i$ will be enumerated thus

$\forall i = 0 \text{ to } 7$ as shown below

$$\forall i=0, (15-0)! \binom{7}{0} \bar{h}^0 = 15!$$

$$\forall i=1, (15-1)! \binom{7}{1} \bar{h}^1 = (15-1)! \left(\frac{7!}{(7-1)! 1!} \right) \bar{h}^1 = 14! 7\bar{h}^1 \quad \forall i=2, (15-2)! \binom{7}{2} \bar{h}^2 = 13!$$

$$\left(\frac{7!}{(7-2)! 2!} \right) \bar{h}^2 = 13! 21\bar{h}^2 \quad \forall i=3, (15-3)! \binom{7}{3} \bar{h}^3 = 12! \left(\frac{7!}{(7-3)! 3!} \right) \bar{h}^3 = 12! 35\bar{h}^3$$

$$\forall i=4, (15-4)! \binom{7}{4} \bar{h}^4 = \frac{7!}{4!(7-4)!} = 11! 35\bar{h}^4 \quad \forall i=5, (15-5)! \binom{7}{5} \bar{h}^5 = \frac{7!}{5!(7-5)!} = 10! 21\bar{h}^5$$

$$\forall i=6, (15-6)! \binom{7}{6} \bar{h}^6 = \frac{7!}{6!(7-6)!} = 9! 7\bar{h}^6 \quad \forall i=7, (15-7)! \binom{7}{7} \bar{h}^7 = 8! \frac{7!}{7!(7-7)!} = 8! \bar{h}^7$$

Such that, $\Sigma \varphi(\bar{h}) = 15! + 14! \binom{7}{1} \bar{h}^1 + 13! \binom{7}{2} \bar{h}^2 + 12! \binom{7}{3} \bar{h}^3 + 11! \binom{7}{4} \bar{h}^4 + 10! \binom{7}{5} \bar{h}^5 + 9! \binom{7}{6} \bar{h}^6 + 8! \binom{7}{7} \bar{h}^7$ (2.3)

From equation (3.2) we have the equation below

$$\Sigma \varphi(\bar{h}) = 15! + 14! 7\bar{h} + 13! 21\bar{h}^2 + 12! 35\bar{h}^3 + 11! 35\bar{h}^4 + 10! 21\bar{h}^5 + 9! 7\bar{h}^6 + 8! \bar{h}^7$$
 (2.4)

Using the same process of binomial expansion to solve equation (3.3) we have the following below.

$\forall i = 0$ to 3 as shown below

$$\forall i = 0, (7 - 0)! \binom{3}{0} \eta^0 = 7!$$

$$\forall i = 1, (7 - 1)! \binom{3}{1} \eta^1 = (7 - 1)! \left(\frac{3!}{(3 - 1)!1!} \right) \eta^1 = 6!3\eta^1$$

$$\forall i = 2, (7 - 2)! \binom{3}{2} \eta^2 = (7 - 2)! \left(\frac{3!}{(3 - 2)!2!} \right) \eta^2 = 5!3\eta^2$$

$$\forall i = 3, (7 - 3)! \binom{3}{3} \eta^3 = (7 - 3)! \left(\frac{3!}{(3 - 3)!3!} \right) \eta^3 = 4!\eta^3$$

To produce the real part and the imaginary part of equation above note that $\eta = u + iv$, by expanding using binomial theorem

$$\begin{aligned} \phi(\eta) &= \sum_{i=0}^3 (2k - 1 - i)! \binom{k-1}{i} \eta^i = \sum_{i=0}^3 (7 - 1)! \binom{3}{i} \eta^i = 7! + 6!3(u + iv) + 5!3(u + iv)^2 + 4!(u + iv)^3 \\ &= 7! + 6!3(u + iv) + 5!3(u^2 - v^2 + i(2uv)) + 4!(u^3 - 3uv^2 + i(3vu^2 - v^3)) \quad (2,5) \end{aligned}$$

Using the same method to solve the denominator we have the following

$$\phi(\eta) = \sum_{i=0}^k (-1)^i (2k - 1 - i)! (\eta^i)$$

hence

$$\phi(\eta) = \sum_{i=0}^4 (-1)^i (7 - i)! \binom{4}{i} \eta^i$$

$$\forall i = 0, (-1)^0 (7 - 0)! \left(\frac{7!}{(7 - 0)!0!} \right) \eta^0 = 7!$$

$$\forall i = 1, (-1)^1 (7 - 1)! \left(\frac{7!}{(7 - 1)!1!} \right) \eta^1 = -6!7\eta^1$$

$$\forall i = 2, (-1)^2 (7 - 2)! \left(\frac{7!}{(7 - 2)!2!} \right) \eta^2 = 5!21\eta^2$$

$$\forall i = 3, (-1)^3 (7 - 3)! \left(\frac{7!}{(7 - 3)!3!} \right) \eta^3 = -4!35\eta^3$$

$$\forall i = 4, (-1)^4 (7 - 4)! \left(\frac{7!}{(7 - 4)!4!} \right) \eta^4 = 3!35\eta^4$$

The procedure above will generate the expansion of the denominator as follows

$$\psi(\eta) = \sum_{i=0}^k (-1)!(2k-1-i)! \binom{k}{i} \eta^i = 7!-6!7\eta^1 + 5!21\eta^2 - 4!35\eta^3 + 3!35\eta^4 \quad (2.6)$$

Expanding by binomial theorem in the denominator part

$$\begin{aligned} \sum_{i=0}^k (-1)!(2k-1-i)! \binom{k}{i} \eta^i &= 7!-6!7(u+iv) + 5!21(u^2-v^2+i(2uv)) - 4!35(u^3-3uv^2+i(3vu^2-u)) \\ &\quad + 3!35(u^4+v^4-6u^2v^2+i(4vu^3-4uv^3)) \end{aligned}$$

(2.7)

We adopt the principle of binomial theorem and that of De-Moivre theorem in equation (2.5) of numerator part and the denominator part of equation (2.7) above, where we substitute

$u = r \cos \phi$ and $v = ir \sin \phi$ from the numerator part of the equation (2.5) and (2.7) we have

$$\begin{aligned} \phi(\eta) &= \sum_{i=0}^{k-1} (2k-1-i)! \binom{k-1}{i} \eta^i = 7!-6!3(r \cos \theta + ir \sin \theta) + 5!3(r^2 \cos^2 \theta + 2r^2 \cos \theta i \sin \theta - r^2 \sin^2 \theta) \\ &\quad + 4!(r^3 \cos^3 \theta + 3r^2 \cos^2 \theta i \sin \theta - 3r^2 \cos \theta \sin^2 \theta - ir^3 \sin^3 \theta) \end{aligned}$$

(2.8)

also

$$\begin{aligned} \psi(\eta) &= \sum_{i=0}^k (-1)!(2k-1-i)! \binom{k}{i} \eta^i = 7!-6!7(r \cos \theta + ir \sin \theta) + 5!21(r^2 \cos^2 \theta + 2r^2 \cos \theta i \sin \theta - r^2 \sin^2 \theta) \\ &\quad - 4!35(r^3 \cos^3 \theta + 3r^3 \cos^2 \theta i \sin \theta - 3r^3 \cos \theta \sin^2 \theta - ir^3 \sin^3 \theta) + 3!35(r^4 \cos^4 \theta + 4r^4 \cos^3 \theta i \sin \theta \\ &\quad - 6r^4 \cos \theta \sin^2 \theta - 4r^4 \cos \theta i \sin^3 \theta - r^4 \sin^4 \theta) \end{aligned}$$

(2.9)

To get the Jordan curve, we define a function $F(\theta, R)$ in respect of $\phi(\bar{h})$ and $\psi(\bar{h})$ with the transformation that $\phi(\bar{h}) \cdot = \phi(\theta, R) \cdot$ and $\psi(\bar{h}) = \psi(\theta, R)$ which gives

$F(\theta, R) = |\phi(\theta, R)|^2 - |\psi(\theta, R)|^2$ for which the RAS is defined in respect of (θ, R) from equation (2.8) and (2.9) we have the equation below

$$\begin{aligned} \phi - \psi = |\phi|^2 - |\psi|^2 = & [2 \times 15! 14! 7\cos\theta + 2 \times 15! 14! 8\cos\theta]R + \{[(14! 7)^2 + 2 \times 15! 13! \\ & 28\cos 2\theta] - (14! 8)^2 - 2 \times 15! 13! 28\cos 2\theta\}R^2 + \{[2 \times 15! 12! 35\cos 3\theta + 14 14! 13! 21\cos\theta] + 2 \times \\ & 15! 12! 56\cos 3\theta + 16 \times 14! 13! 28\cos\theta\}R^3 + \{(13! 21)^2 + 2 \times 15! 11! 35\cos 4\theta + 14 \times 14! 12! \\ & 35\cos 2\theta - (13! 28)^2 - 2 \times 15! 11! 70\cos 4\theta - 16 \times 14! 12! 56\cos 2\theta\}R^4 + \{2 \times 15! 10! 21\cos 5\theta + \\ & 14! 11! 35\cos 3\theta + 42 \times 13! 12! 36\cos\theta\}R^5 + \{(12! 56)^2 + 2 \times 15! 9! 7\cos 6\theta + 14 \times 14! 10! \\ & 21\cos 4\theta + 42 \times 13! 11! 35\cos\theta - (12! 56)^2 - 2 \times 15! 9! 28\cos 6\theta - 16 \times 14! 10! 56\cos 4\theta - 56 \times 13! \\ & 11! 70\cos 2\theta\}R^6 + \{2 \times 15! 8! \cos 7\theta + 14 \times 14! 7\cos 5\theta + 42 \times 13! 10! 21\cos 3\theta + 70 \times 12! 11! \\ & 35\cos\theta + 2 \times 15! 8! 8\cos 7\theta + 16 \times 14! 9! 28\cos 5\theta + 56 \times 13! 10! 56\cos 3\theta + 112 \times 12! 11! \\ & 70\cos\theta\}R^7 + \{(11! 35)^2 + 14 \times 14! 8\cos 6\theta + 42 \times 13! 13! 9! 7\cos 4\theta + 70 \times 12! 10! 21\cos 2\theta - (11! \\ & 70)^2 - 2 \times 15! 7! \cos 8\theta - 16 \times 14! 8! \cos 6\theta - 56 \times 13! 9! 28\cos 4\theta + 112 \times 12! 10! 56\cos 2\theta\}R^8 + \\ & \{42 \times 13! 8! \cos 5\theta + 70 \times 12! 9! 7\cos 3\theta + 70 \times 11! 10! 21\cos\theta + 16 \times 14! 7! \cos 7\theta + 56 \times 13! 8! \\ & \cos 5\theta + 112 \times 12! 9! 28\cos 3\theta + 140 \times 11! 10! 56\cos\theta\}R^9 + \{(10! 21)^2 + 70 \times 12! 8! \cos 4\theta + 70 \times \\ & 11! 9! 7\cos 2\theta - (10! 56)^2 - 56 \times 13! 7! \cos 6\theta - 12 \times 12! 8! 8! \cos 4\theta - 140 \times 11! 9! 28\cos 2\theta\}R^{10} + \\ & \{70 \times 11! 8! \cos 3\theta + 42 \times 10! 9! \cos\theta + 112 \times 12! 7! \cos 5\theta + 140 \times 11! 8! \cos 3\theta + 112 \times 10! 9! \\ & 28\cos\theta\}R^{11} + \{(9! 7)^2 + 42 \times 10! 8! \cos 2\theta - (9! 28)^2 - 140 \times 11! 7! \cos 4\theta - 112 \times 10! 8! \\ & 8! \cos 2\theta\}R^{12} + \{14 \times 9! 8! \cos\theta + 112 \times 10! 7! \cos 3\theta + 56 \times 9! 8! \cos\theta\}R^{13} + (8!)^2 - \{(8! 8)^2 - 56 \times \\ & 9! 7! \cos 2\theta\}R^{14} + (16 \times 8! 7! \cos\theta\}R^{15} + \\ & \{-(7!)^2\}R^{16} \quad (2.10) \end{aligned}$$

Therefore

$$|\phi|^2 - |\psi|^2 = 2(15!)^2 \cos \theta R - 15! 14! (1 + \cos 2\theta) R^2 + (14!)^2 (15 \cos 3\theta + 53 \cos \theta) R^3 - 13! 12! (4459 + 1225 \cos 4\theta + 5684 \cos 2\theta) R^4 + 2 \times 13! 11! (1470 \cos 5\theta + 11270 \cos 3\theta + 27636 \cos \theta) R^5 - 14! 10! (1386 + 63 \cos 6\theta + 60 + 602 \cos 4\theta + 1925 \cos 2\theta) R^6 + 2 \times 12! 9! (2730 \cos 7\theta + 49686 \cos 5\theta + 261170 \cos 3\theta + 56595 \cos \theta) R^7 + 2 \times 11! 8! (1819125 + 4095 \cos 8\theta + 2184 \cos 6\theta + 894348 \cos 4\theta + 2593080 \cos 2\theta) R^8 + 2 \times 11! 8! (2184 \cos 7\theta + 38220 \cos 5\theta + 195804 \cos 3\theta + 418950 \cos \theta) R^9 - 2 \times 11! 8! (11025 + 546 \cos 6\theta + 4956 \cos 4\theta + 15435 \cos 2\theta) R^{10} + 2 \times 10! 8! (924 \cos 5\theta + 6545 \cos 3\theta + 15435 \cos \theta) R^{11} - 2 \times 9! 7! (26460 + 7700 \cos 4\theta + 334160 \cos \theta) R^{12} + 2 \times 9! 8! (70 \cos 3\theta + 231 \cos \theta) R^{13} - 9! 8! 7(1 + \cos 2\theta) R^{14} + 8! 7! (16 \cos 2\theta) R^{15} - (7!) R^{16} \quad (2.11)$$

Since R is common to all terms, we divide through by R and have the result in the absolute value.

The whole expression in above equation (2.11) can then be equated to zero. Hence $R = 0$ implies that

R has a root whose value is zero. Observe that the rational integrator in now is linear inequality form. To continue the solving we set $F(\theta, R) = 0$.

Also observe that all even power of R take positive sign while odd power of R take negative sign.

This happened after a complete factorization of all the terms in R has taken place. We obtain the

Jordan curve for $k = 8$

The Jordan curve is given by $F(\theta, R) = |\phi|^2 - |\psi|^2 \leq 0$ is now written in the equation below

$$F(\theta, R) = |\phi|^2 - |\psi|^2 = 2(15!)^2 \cos \theta - 15! 14! (1 + \cos 2\theta) R + (14!)^2 (15 \cos 3\theta + 53 \cos \theta) R^2 - 13! 12! (4459 + 1225 \cos 4\theta + 5684 \cos 2\theta) R^3 + 2 (13! 11!) (1470 \cos 5\theta + 11270 \cos 3\theta + 27636 \cos \theta) R^4 - 14! 10! (1386 + 63 \cos 4\theta + 1925 \cos 2\theta) R^5 + 2 \times 12! 9! (2730 \cos 7\theta + 49686 \cos 5\theta +$$

$$261170\cos 3\theta + 565950\cos \theta)R^6 - 2 \times 11! \cdot 8! (1819125 + 4095\cos 8\theta + 124488\cos 6\theta + 894348\cos 4\theta + 2593080\cos 2\theta)R^7 + 2 \times 11! \cdot 8! (2184\cos 7\theta + 38220\cos 3\theta + 418950\cos \theta)R^8 - 2 \times 11! \cdot 8! (11025 + 546\cos 6\theta + 4956\cos 4\theta + 15435\cos \theta)R^9 + 2 \times 10! \cdot 8! (924\cos 5\theta + 6545\cos 3\theta + 15435\cos \theta)R^{10} - 2 \times 9! \cdot 7! (26460 + 7700\cos 4\theta + 334160\cos 2\theta)R^{11} + 2 \times 9! \cdot 8! (70\cos 3\theta + 231\cos \theta)R^{12} - 9! \cdot 8! \cdot 7(1 + \cos 2\theta)R^{13} + 8! \cdot 7! (16\cos \theta)R^{14} - (7!)R^{15} = 0 \quad (2.12)$$

The above equation (2.12) is the required equation for the integrator. So, from the above equation (2.12) is observed that the highest power of R is $-R^{15}$. Therefore, there exist a real $\tau > 0$ for which

$$F(\theta, R) = 0.$$

To find that number we first construct a trigonometrically table in terms of $F(\theta, R)$ that will evaluate the value of $\cos \theta$ in the equation for $0^0 < \theta < 2\pi$ in the intervals of 30^0 . The equation $|S(\eta)|=1$ define the boundary of the Jordan curve which is between the Region of Absolute Stability and the Region of instability of the function $F(\theta, R)$.

So, from the above equation (2.12) will construct values for cost in terms of R, which we eventually use to evaluate R to enable us to draw the graph of the Jordan curve. So substituting each of the degree into the equation (2.12) from $0^0 < \theta < 2\pi$ in the table below

Table 1: Computation of F (θ,R) for $0^0 < \theta < 2\pi$.

So from the above equation (2.12)

Radiant	F(θ,R)
0	$3.420024505 \text{ E}24 - 2.280016337 \text{ E}23R + 5.16803703 \text{ E}23R^2 - 3.390793527 \text{ E}22R^2 +$

	$2.007193869 E22R^2 - 1.25781787 E21R^5 + 3.05762072 E20R^6 - 1.749510901 E19R^7 + 2.108882027 E18R^8 + 1.028821862 E17R^9 + 6.70231579915R^{10} - 2.499029729 E14R^{11} + 8.8808055603 E12R^{12} - 2.048385024 E11R^{13} + 3251404800R^{14} - 25401600R^{15} \leq 0$
$\frac{\pi}{6}$	$2.961828103 E24 - 1.710012253 E23R + 3.488375322 E23R^2 - 1.995014295 E22R^3 + 1.12650727 E22R^4 - 6.278017012 E20R^5 + 1.546072399 E20R^6 - 8.182277051 E18R^7 + 1.055250412 E18R^8 - 5.059613429 E16R^9 + 3.677405538 E15R^{10} - 1.451792886 E14R^{11} + 5.854046443 E12R^{12} - 1.536288768 E11R^{13} + 2815799155R^{14} - 25401600R^{15} \leq 0$
$\frac{\pi}{4}$	$2.41832252 E24 - 1.140008168 E23R + 2.042139017 E23R^2 - 0646222964 E21R^3 + 5.236253889 E21R^4 - 2.480204252 E20R^5 + 6.337797178 E19R^6 - 2.989937491 E18R^7 + 4.258809734 E17R^8 - 1.953544797 E16R^9 + 1.643809859 E15R^{10} - 6.86208983 E13R^{11} + 3.331381964 E12R^{12} - 1.024192512 E11R^{13} + 2299090382R^{14} - 25401600R^{15} \leq 0$
$\frac{\pi}{3}$	$1.710012253 E24 - 5.70040842 E22R + 8.740062625 E22R^2 - 2.996175315 E21R^3 + 1.632062976 E21R^4 - 5.868340417 E19R^5 + 1.669124898 E19R^6 - 6.368623636 E17R^7 + 1.090334864 E17R^8 - 4.427584229 E15R^9 + 4.782979031 E14R^{10} - 1.451792886 E14R^{11} + 1.331450266 E12R^{12} - 5.12096256 E10R^{13} + 1625702400R^{14} - 25401600R^{15} \leq 0$
$\frac{\pi}{2}$	$-25401600R^{15} \leq 0, \quad F(90^0, 0), \quad R = 0$

$\frac{2\pi}{3}$	$1.710012253 E24 - 5.70040842 E22R + 8.740062625 E22R^2 - 2.996175315 E21R^3 +$ $1.632062976 E21R^4 - 5.868340417 E19R^5 + 1.669124898 E19R^6 - 6.368623636$ $E17R^7 + 1.090334864 E17R^8 - 4.427584229 E15R^9 + 4.782979031 E14R^{10} -$ $1.451792886 E14R^{11} + 1.331450266 E12R^{12} - 5.12096256 E10R^{13} + 1625702400R^{14} -$ $25401600R^{15} \leq 0$
$\frac{3\pi}{4}$	$-2.961828103 E24 - 1.71012253 E23R - 3.488375322 E23R^2 - 1.995014295E22R^3 -$ $1.12650727 E22R^4 - 6.278017012 E20R^5 - 1.546072399 E20R^6 - 8.182277051 E18R^7$ $- 1.055250412 E18R^8 - 5.059613429 E16R^9 - 3.677405538 E15R^{10} - 1.451792886$ $E14R^{11} - 5.854046443E^{12} - 1.536288768 E11R^{13} - 2815799155R^{14} - 25401600R^{15}$ ≤ 0
$\frac{5\pi}{6}$	$3.420024505 E24 - 2.280016337 E23R + 5.16803703 E23R^2 - 3.390793527 E22R^2 +$ $2.007193869 E22R^4 - 1.25781787 E21R^5 + 3.05762072 E20R^6 - 1.749510901 E19R^7$ $+ 2.108882027 E18R^8 - 1.028821862 E17R^9 + 6.70231579915R^{10} - 2.499029729$ $E14R^{11} + 8.8808055603 E12R^{12} - 2.048385024 E11R^{13} + 3251404800R^{14} -$ $25401600R^{15} \leq 0$
Π	$2.961828103 E24 - 1.71012253 E23R + 3.488375322 E23R^2 - 1.995014295E22R^3 +$ $1.12650727 E22R^4 - 6.278017012 E20R^5 + 1.546072399 E20R^6 - 8.182277051 E18R^7$ $+ 1.055250412 E18R^8 - 5.059613429 E16R^9 + 3.677405538 E15R^{10} - 1.451792886$ $E14R^{11} + 5.854046443E^{12} - 1.536288768 E11R^{13} + 2815799155R^{14} - 25401600R^{15}$ ≤ 0

$\frac{7\pi}{6}$	$1.710012253 E24 - 5.70040842 E22R + 8.740062625 E22R^2 - 2.996175315 E21R^3 +$ $1.632062976 E21R^4 - 5.868340417 E19R^5 + 1.669124898 E19R^6 - 6.368623636$ $E17R^7 + 1.090334864 E17R^8 - 4.427584229 E15R^9 + 4.782979031 E14R^{10} -$ $1.451792886 E14R^{11} + 1.331450266 E12R^{12} - 5.12096256 E10R^{13} + 1625702400R^{14} -$ $25401600R^{15}$ ≤ 0
$\frac{5\pi}{4}$	$25401600R^{15} \leq 0 \text{ F}(270^0, 0), R = 0$
$\frac{4\pi}{3}$	$1.710012253 E24 - 5.70040842 E22R + 8.740062625 E22R^2 - 2.996175315 E21R^3 +$ $1.632062976 E21R^4 - 5.868340417 E19R^5 + 1.669124898 E19R^6 - 6.368623636$ $E17R^7 + 1.090334864 E17R^8 - 4.427584229 E15R^9 + 4.782979031 E14R^{10} -$ $2.02278011 E13R^{11} + 1.331450266 E12R^{12} - 5.12096256 E10R^{13} + 1625702400R^{14} -$ $25401600R^{15} \leq 0$
$\frac{3\pi}{2}$	$-2.961828103 E24 - 1.71012253 E23R - 3.488375322 E23R^2 - 1.995014295E22R^3 -$ $1.12650727 E22R^4 - 6.278017012 E20R^5 - 1.546072399 E20R^6 - 8.182277051 E18R^7$ $- 1.055250412 E18R^8 - 5.059613429 E16R^9 - 3.677405538 E15R^{10} - 1.451792886$ $E14R^{11} - 5.854046443E^{12} - 1.536288768 E11R^{13} - 2815799155R^{14} - 25401600R^{15}$ ≤ 0
$\frac{5\pi}{3}$	$3.420024505 E24 - 2.280016337 E23R + 5.16803703 E23R^2 - 3.390793527 E22R^2 +$ $2.007193869 E22R^2 - 1.25781787 E21R^5 + 3.05762072 E20R^6 - 1.749510901 E19R^7$ $+ 2.108882027 E18R^8 1.028821862 E17R^9 + 6.70231579915R^{10} - 2.499029729$

	$E14R^{11} + 8.8808055603 E12R^{12} - 2.048385024 E11R^{13} + 3251404800R^{14} - 25401600R^{15} \leq 0$
$\frac{11\pi}{6}$	$\begin{aligned} & -2.961828103 E24 - 1.71012253 E23R - 3.488375322 E23R^2 - 1.995014295E22R^3 - \\ & 1.12650727 E22R^4 - 6.278017012 E20R^5 - 1.546072399 E20R^6 - 8.182277051 E18R^7 \\ & - 1.055250412 E18R^8 - 5.059613429 E16R^9 - 3.677405538 E15R^{10} - 1.451792886 \\ & E14R^{11} - 5.854046443E^{12} - 1.536288768 E11R^{13} - 2815799155R^{14} - 25401600R^{15} \\ & \leq 0 \end{aligned}$
2π	$\begin{aligned} & 3.420024505 E24 - 2.280016337 E23R + 5.16803703 E23R^2 - 3.390793527 E22R^2 + \\ & 2.007193869 E22R^2 - 1.25781787 E21R^5 + 3.05762072 E20R^6 - 1.749510901 E19R^7 \\ & + 2.108882027 E18R^8 1.028821862 E17R^9 + 6.70231579915R^{10} - 2.499029729 \\ & E14R^{11} + 8.8808055603 E12R^{12} - 2.048385024 E11R^{13} + 3251404800R^{14} - \\ & 25401600R^{15} \leq 0 \end{aligned}$

Observation $F(\theta, R) = 0$ at $\theta = 90^0$ and 270^0 , $R = 0$ the Jordan curve for the integrator $k = 8$ lies at the boundary.

Table 2: Summary of Table 1 above

Radiant	$\tau = F(\tau, 0)$	a_n	b_n	Minimum interval size $b_n - a_n$
0	61.48147892	61.48147691	61.48148092	0.00000401

$\frac{\pi}{6}$	56.14748211	56.14746975	56.14749446	0.00002471
$\frac{\pi}{4}$	49.57874854	49.57872843	49.57876864	0.00004021
$\frac{\pi}{3}$	12.21453484	12.21482349	12.21484629	0.00002280
$\frac{\pi}{2}$	0.00000000	0.00000000	0.00000000	0.00000000
$\frac{2\pi}{3}$	-12.21483489	12.21482349	12.21484629	0.00002280
$\frac{3\pi}{4}$	-49.57874834	49.57872843	49.57876864	0.00004021
$\frac{5\pi}{6}$	-56.14748211	56.14746975	56.14749446	0.00002471
π	-61.48147892	61.48147691	61.48148092	0.00000401
$\frac{7\pi}{6}$	-56.14748211	56.14746975	56.14749446	0.00002471
$\frac{5\pi}{4}$	-49.57874854	49.578728443	49.57876864	0.00004021
$\frac{4\pi}{3}$	-12.21483489	12.214823349	12.211484629	0.00002280
$\frac{3\pi}{2}$	0.00000000	0.00000000	0.00000000	0.00000000
$\frac{5\pi}{3}$	12.21483489	12.21482349	12.21484629	0.00002280
$\frac{7\pi}{4}$	49.57874854	49.57872843	49.57876864	0.00004021
$\frac{11\pi}{6}$	56.14748211	56.14746975	56.14749446	0.00002471

2π	61.48147892	61.48147691	61.48148092	0.00000401
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Observe $F(0,R) = 0$ at $\theta = 90^\circ$ and 270° $R = 0$, the Jordan curves for the integrator $k = 8$ lies the boundary.

The summary of the result shall be given because of the lengthy nature and the repetition of the angels with changes in signs.

Table 3: Summary of Results in Table 2

Radiant	Evaluated to 8 places of decimal	approximate values for the Curve Sketch
0	61.48147892	61.48
$\frac{\pi}{6}$	56.14748211	56.15
$\frac{\pi}{4}$	49.57874854	49.58
$\frac{\pi}{3}$	12.21453484	12.21
$\frac{\pi}{2}$	0.00000000	0.00000000
$\frac{2\pi}{3}$	-12.21483489	-12.21
$\frac{3\pi}{4}$	-49.57874834	-49.58

$\frac{5\pi}{6}$	-56.14748211	-56.15
π	-61.48147892	-61.48
$\frac{7\pi}{6}$	-56.14748211	-56.15
$\frac{5\pi}{4}$	-49.57874854	-49.58
$\frac{4\pi}{3}$	-12.21483489	-12.21
$\frac{3\pi}{2}$	0.00000000	0.00000000
$\frac{5\pi}{3}$	12.21483489	12.21
$\frac{11\pi}{6}$	56.14748211	56.15
2π	61.48147892	61.4814789

We shall use the approximated values for $F(\theta, R)$ to sketch the graph of the Jordan Curve to show the Region of Absolute Stability (RAS) for $k = 8$ as shown in the graph below

Table 4: Comparismof Result

Radiant	K = 3 T	K = 4 T	K = 5 T	K = 7 T	K=8
0	11.8	1.2	27.8	87.6	61.5
$\frac{\pi}{6}$		17.7	25.6	69.0	56.2
$\frac{\pi}{4}$		16.0	23.0	40.0	49.6
$\frac{\pi}{3}$	8.6	13.5	19.1	22.0	12.2
$\frac{\pi}{2}$	0	0	0	0	0
$\frac{2\pi}{3}$	8.6	13.5	19.1	22.0	- 12.2
$\frac{5\pi}{6}$		17.7	25.6	69.0	- 56.2
Π	11.9	19.8	27.8	87.6	- 61.5
$\frac{7\pi}{6}$		17.7	25.6	69.0	- 56.2
$\frac{4\pi}{3}$	8.6	-13.5	-19.1	-22.0	- 12.2
$\frac{3\pi}{2}$	0	0	0	0	0
$\frac{5\pi}{3}$	8.6	13.5	19.1	22.0	12.2
$\frac{11\pi}{6}$		17.7	25.6	69.0	56.2

2π	11.8	19.2	27.8	87.6	61.5
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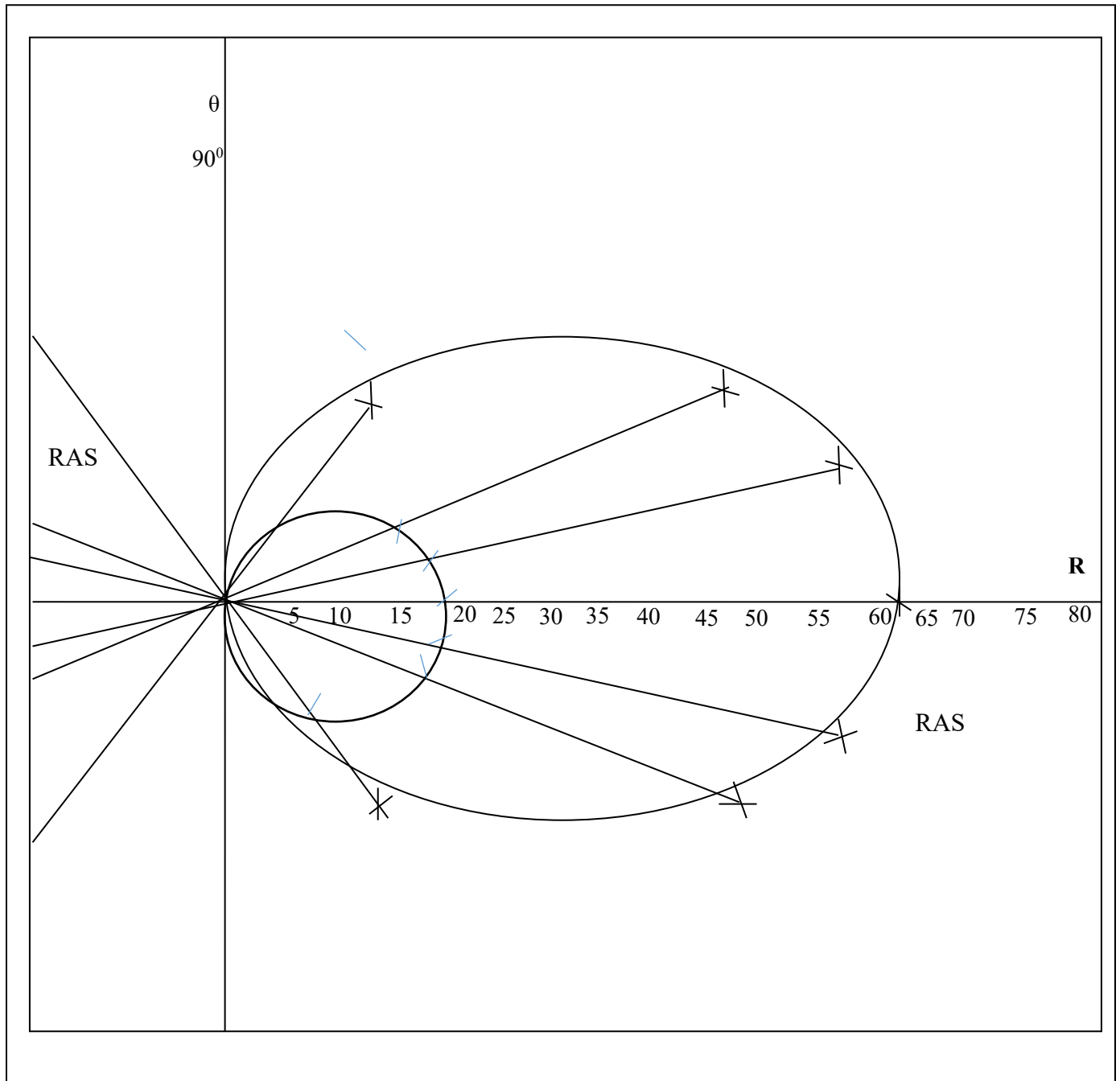


Figure 1: A Sketch Graph of the Jordan Curve Showing Region of Absolute Stability (RAS)

for the Case $K = 8$

3. Results

After a successful expansion of the given rational integrator, it becomes possible to represent the expanded function in Jordan form. Here, the results shows the region of absolute stability ,exterior (outside) the curved of the Jordan curve represent the region of absolute stability (RAS) of the 15th and 7th order integrator, the RIS is on the positive side of the complex plane, which the region of absolute stability was outside the curve and the rational integrator is within the internal - $61.48147892 < \tau < 61.48148092$ and $-19.2 < \tau < 19.2$ where the τ is the encroachment point.

4. Conclusion

It can be concluded that the Region of Absolute Stability (RAS) of the rational Interpolation method lies entirely on the Left-half of the complex plane and at the same time, the region of instability (RIS). This work revealed that, the Region of Absolute Stability (RAS) for $k = 8$ superset of the entire left-half of the complex plane of $k=4$. It has been shown that the integrator of both are A- stable.

5. References

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